Lecture 11

Dr.
Mahmoud Khedr

Third Edition

CHAPTER

9

MECHANICS OF MATERIALS

Ferdinand P. Beer
E. Russell Johnston, Jr.
John T. DeWolf

Lecture Notes:
J. Walt Oler
Texas Tech University

Deflection of Beams

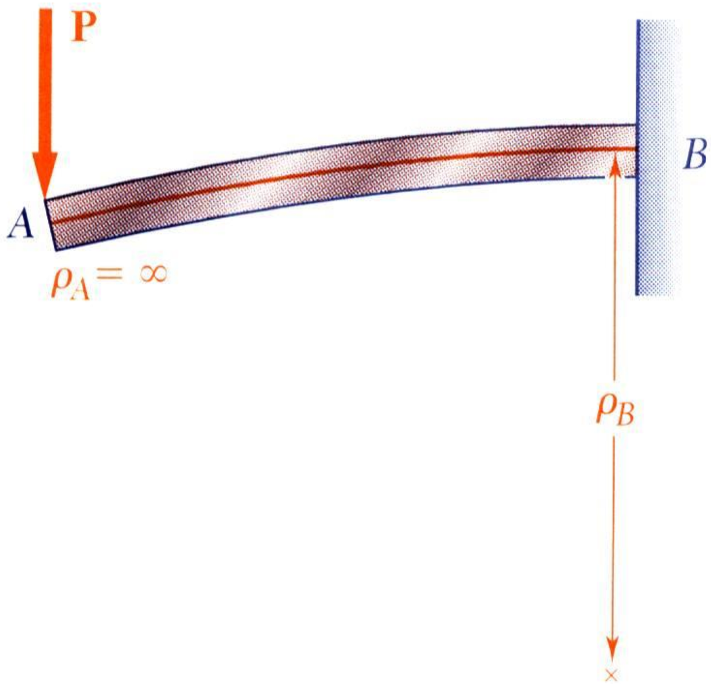
Why should we calculate the deflection for shafts or beams?

- Designing beams or shafts require to satisfy rigidity and strength conditions:

$\theta < \theta_{all}$ $\sigma < \sigma_{all},$ $(y, \theta) < (y, \theta)_{all}$

- Loaded beams or shafts must have limits in the deflection and slope
- Various analytical and semi graphical methods are used to determine the deflection and slope of beams at specific points.

Deformation of a Beam Under Transverse Loading



- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load at the free end,

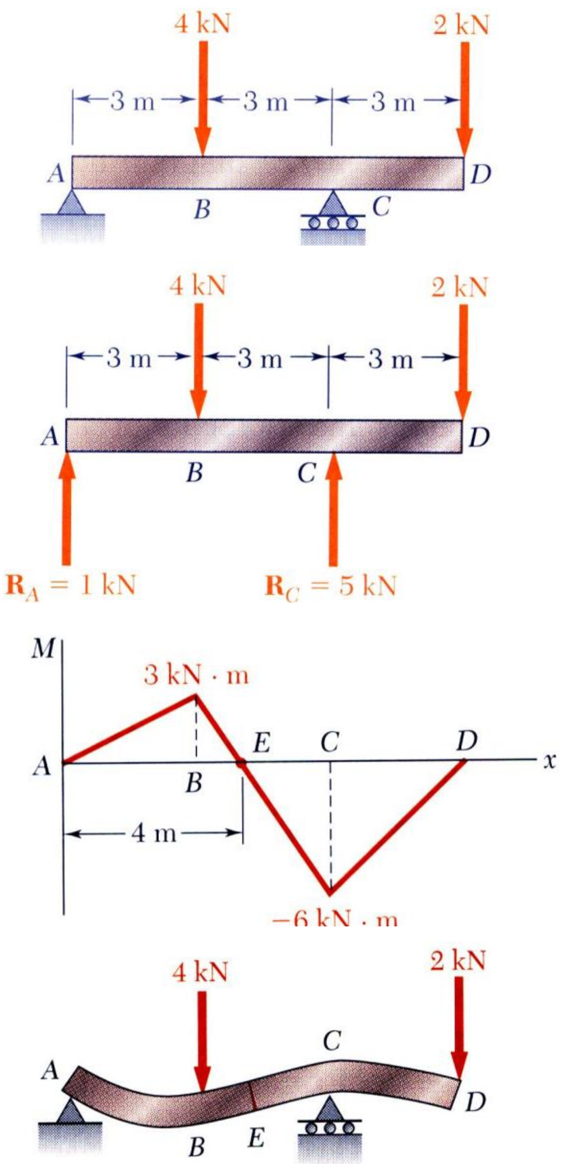
$$\frac{1}{\rho} = -\frac{Px}{EI}$$

- Curvature varies linearly with *x*

- At the free end *A*, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

- At the support *B*, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deformation of a Beam Under Transverse Loading

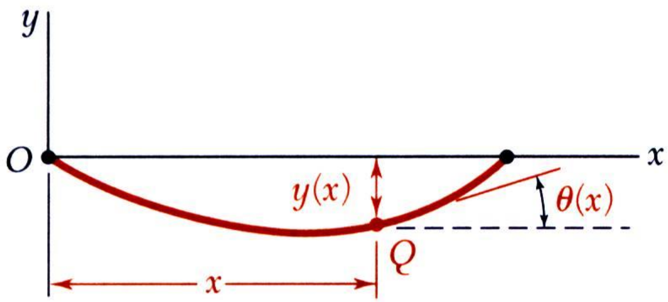


- Overhanging beam
- Reactions at A and C
- Bending moment diagram
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at E.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

Equation of the Elastic Curve



- From elementary calculus, simplified for beam parameters,

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

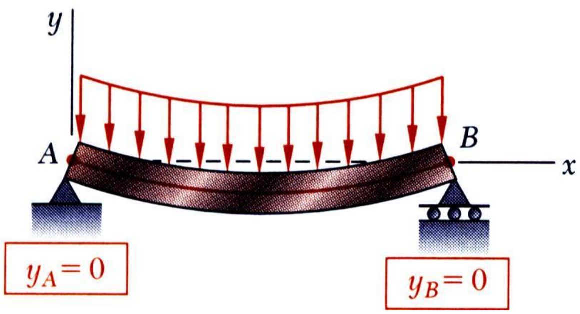
- Substituting and integrating,

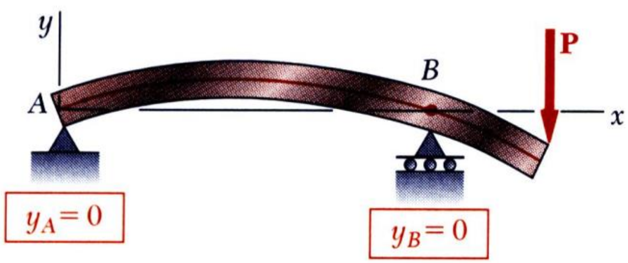
$$EI\frac{1}{\rho} = EI\frac{d^2y}{dx^2} = M(x)$$

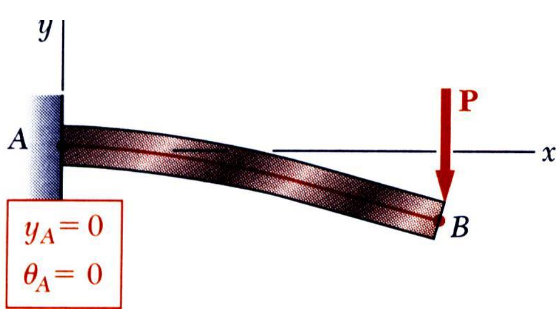
$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)dx + C_1x + C_2$$

Equation of the Elastic Curve







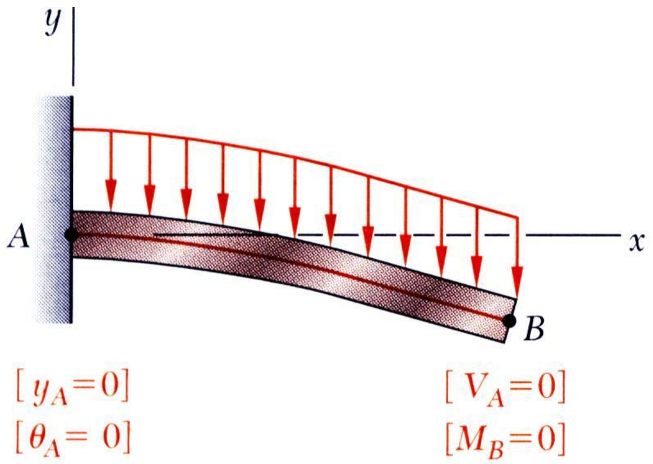
- Constants are determined from boundary conditions

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

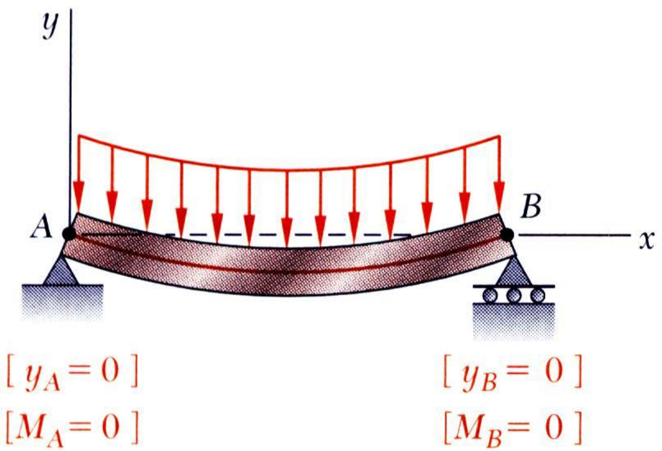
- Three cases for statically determinant beams,
  - Simply supported beam
    $$y_A = 0, \quad y_B = 0$$
  - Overhanging beam
    $$y_A = 0, \quad y_B = 0$$
  - Cantilever beam
    $$y_A = 0, \quad \theta_A = 0$$

- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Direct Determination of the Elastic Curve From the Load Distribution



(*a*) Cantilever beam



(*b*) Simply supported beam

- For a beam subjected to a distributed load,

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

- Equation for beam displacement becomes

$$\frac{d^2M}{dx^2} = EI\,\frac{d^4y}{dx^4} = -w(x)$$

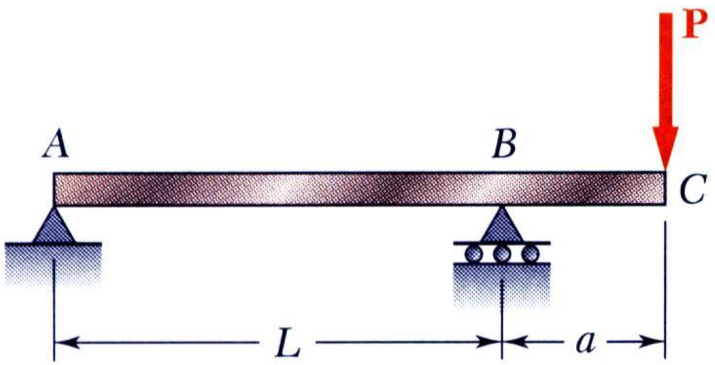
- Integrating four times yields

$$EI\;y(x) = -\int dx \int dx \int dx \int w(x)\,dx$$

$$+\tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

Sample Problem 9.1



$W14 \times 68$ $I = 723\,\text{in}^4$ $E = 29 \times 10^6\,\text{psi}$

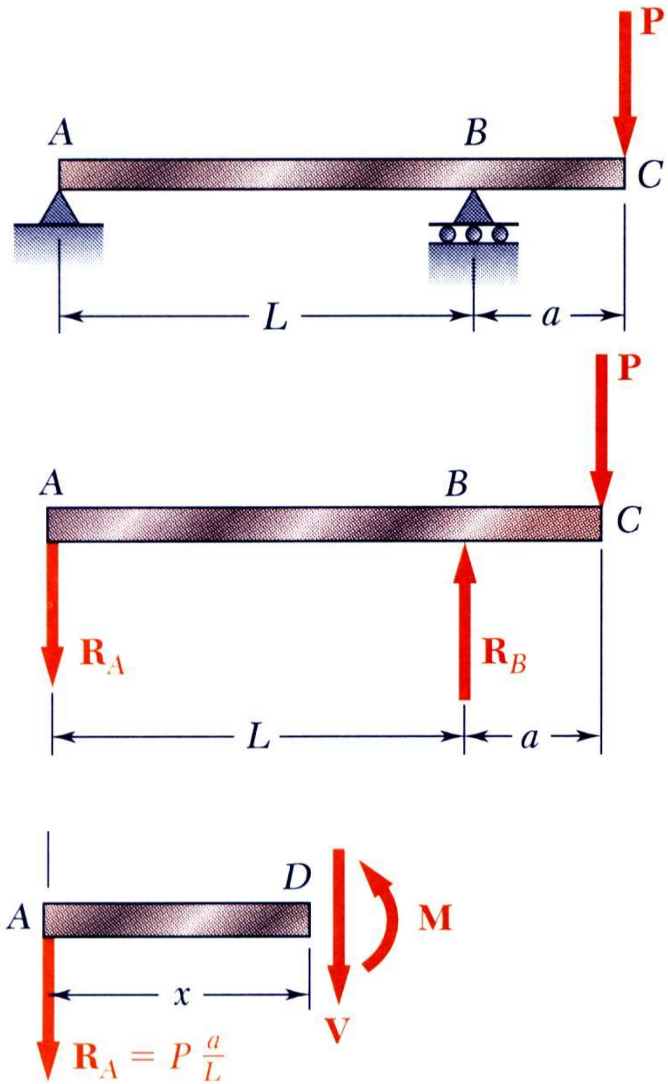
$P = 50\,\text{kips}$ $L = 15\,\text{ft}$ $a = 4\,\text{ft}$

For portion *AB* of the overhanging beam,
(*a*) derive the equation for the elastic curve,
(*b*) determine the maximum deflection,
(*c*) evaluate y_{max}.

SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.
- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.
- Locate point of zero slope or point of maximum deflection.
- Evaluate corresponding maximum deflection.

Sample Problem 9.1



SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.

  - Reactions:

$$R_A = \frac{Pa}{L} \downarrow \qquad R_B = P\left(1 + \frac{a}{L}\right) \uparrow$$

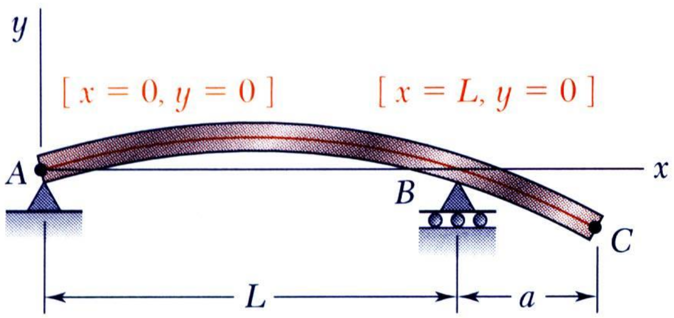
  - From the free-body diagram for section *AD*,

$$M = -P\frac{a}{L}x \qquad (0 < x < L)$$

  - The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Sample Problem 9.1



$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

$$\text{at } x = 0,\ y = 0: \quad C_2 = 0$$

$$\text{at } x = L,\ y = 0: \quad 0 = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \quad C_1 = \frac{1}{6}PaL$$

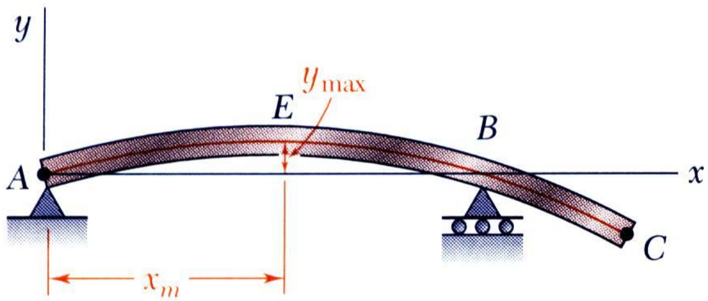
Substituting,

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \quad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx$$

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

Sample Problem 9.1



$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

- Locate point of zero slope or point of maximum deflection.

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

- Evaluate corresponding maximum deflection.

$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right]$$

$$y_{max} = 0.0642\frac{PaL^2}{6EI}$$

$$y_{max} = 0.0642\frac{(50\,\text{kips})(48\,\text{in})(180\,\text{in})^2}{6\left(29\times10^6\,\text{psi}\right)\left(723\,\text{in}^4\right)}$$

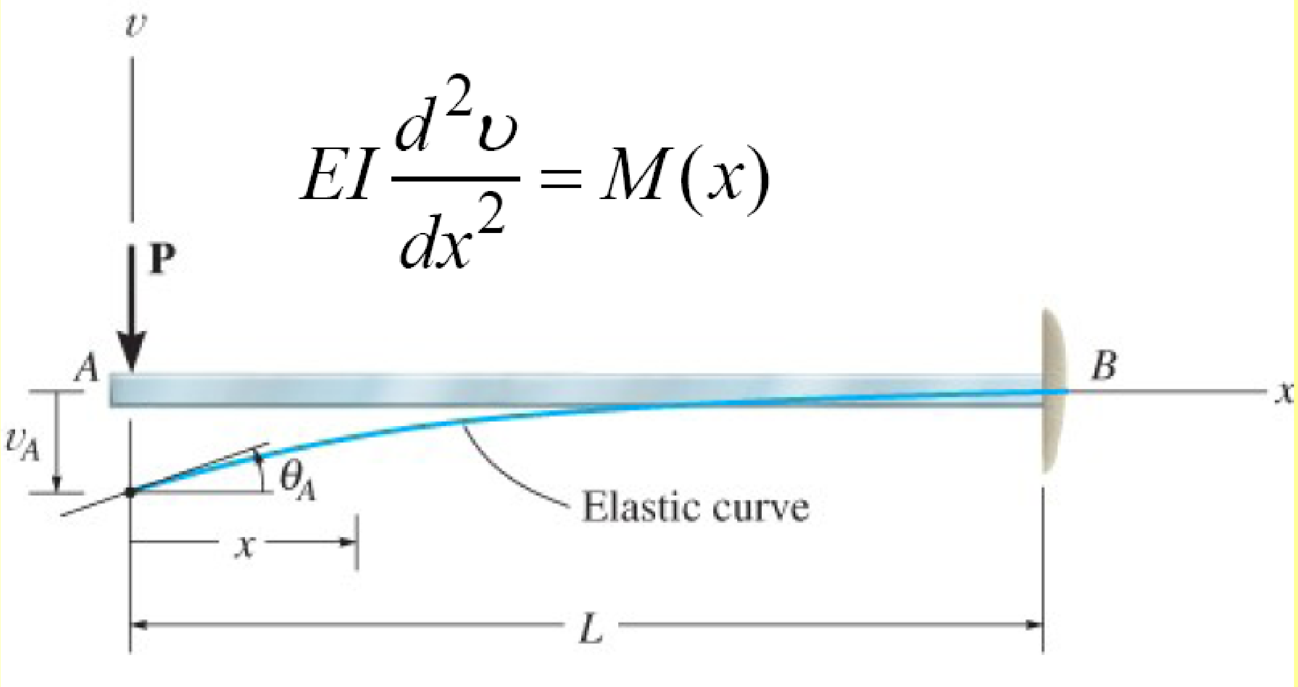
$$y_{max} = 0.238\,\text{in}$$

Example 9.2

Cantilevered beam shown is subjected to a vertical load **P** at its end. **Determine** the **Eqn of the elastic curve (y or v)**. EI is constant.

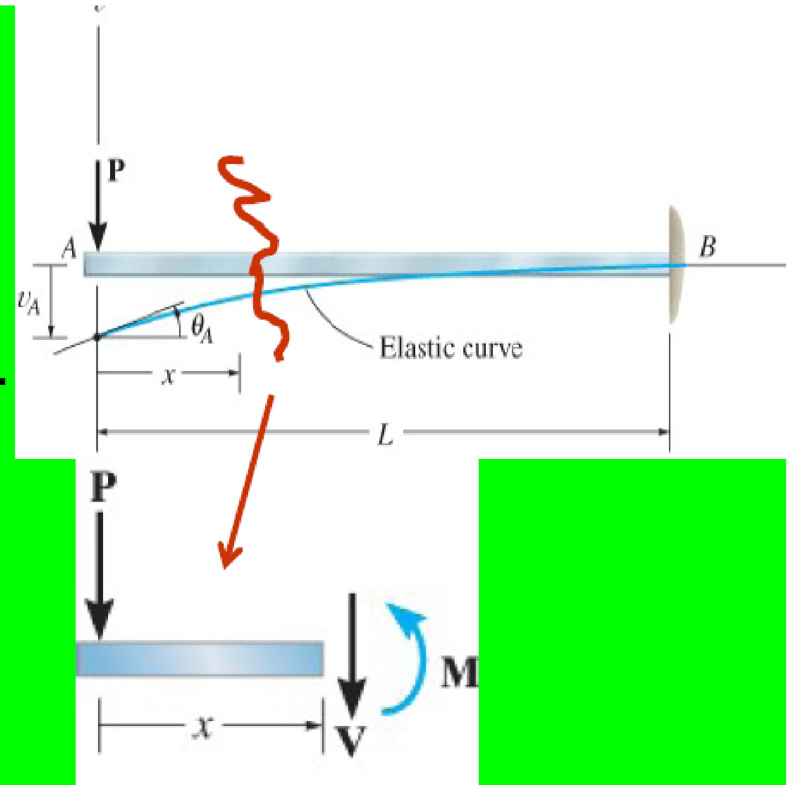
$$y = v$$

$$EI\frac{d^2 v}{dx^2} = M(x)$$



Example 9.2

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$



Elastic curve:

Load tends to deflect the beam.

Moment function:

By inspection, the internal moment can be represented

Throughout the beam using a single x coordinate. From FBD, with **M** acting in the $+ve$ direction, we have

$$M = -Px$$

Example 9.2

$$EI\frac{d^2\upsilon}{dx^2} = M(x) \qquad (12-10), \qquad M(x) = -Px$$

$$EI\frac{d^2\upsilon}{dx^2} = -Px \qquad (1)$$

Slope dv/dx = θ

EI [illegible] $= -\frac{Px^2}{2} + C_1 \qquad (2)$

Displacement, v

EI [illegible] $= -\frac{Px^3}{6} + C_1 x + C_2 \qquad (3)$

Example 9.2

Slope (θ) and elastic curve (v):

Using boundary conditions:

at $x = L$, $dv/dx = 0$ and at $x = L$, $v = 0$,

Eqn (2) and (3) becomes:

$$0 = -\frac{PL^2}{2} + C_1 \quad \Rightarrow\Rightarrow \quad C_1 = \frac{PL^2}{2}$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2 \quad \Rightarrow\Rightarrow \quad C_2 = -\frac{PL^3}{3}$$

Example 9.2

Thus, substituting $C_1 = PL^2/2$ and $C_2 = -PL^3/3$ into Eqns. (2) and (3), yields:

$$\theta = -\frac{P}{2EI}\left(L^2 - x^2\right)$$

$$\upsilon = \frac{P}{6EI}\left(-x^3 + 3L^2x - 2L^3\right)$$

Maximum slope and displacement occur at A $(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \qquad (4) \qquad\qquad \upsilon_A = -\frac{PL^3}{3EI} \qquad (5)$$

Example 9.2

- **Positive result for θ_A** indicates **counterclockwise rotation** and **negative result for A** indicates that v_A is **downward.**
- Consider beam to have a length of 5 m, support load P = 30 kN and made of A-36 steel having E_{st} = 200 GPa.

Assuming allowable normal stress is σ_{allow} = 250 MPa, and (I = 84.8(10^6) mm^4). From Eqns (4) and (5),

$$\theta_A = \frac{PL^2}{2EI} = 0.0221 rad. \qquad \upsilon_A = -\frac{PL^3}{3EI} = -73.3 mm$$

Example 9.2

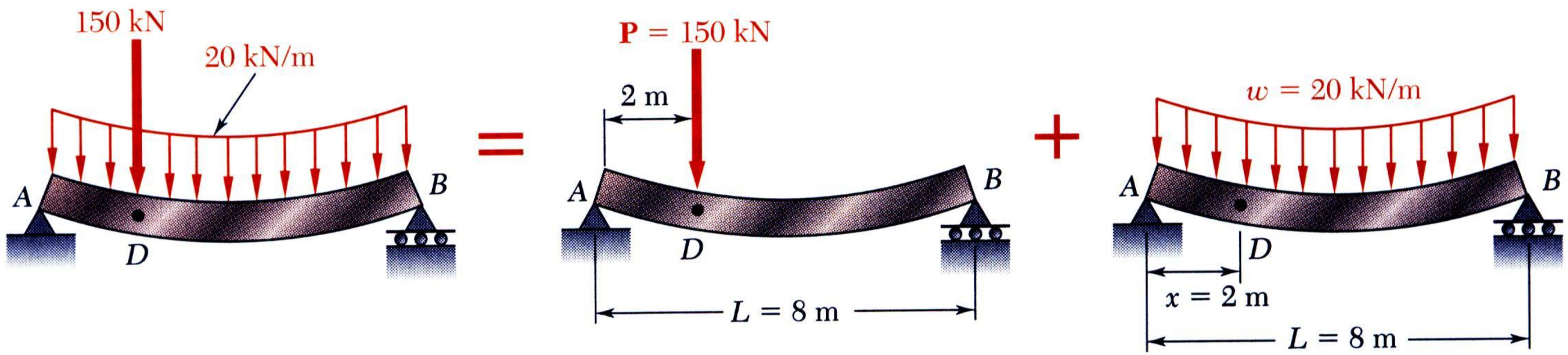
Slope and elastic curve:

From Eqns (4) and (5),

$$\theta_A = \frac{30\ \text{kN}\left(10^3\ \text{N/kN}\right) \times \left[5\ \text{m}\left(10^3\ \text{mm/m}\right)^2\right]^2}{2\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = 0.0221\ \text{rad}$$

$$\upsilon_A = -\frac{30\ \text{kN}\left(10^3\ \text{N/kN}\right) \times \left[5\ \text{m}\left(10^3\ \text{mm/m}\right)^2\right]^3}{3\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = -73.7\ \text{mm}$$

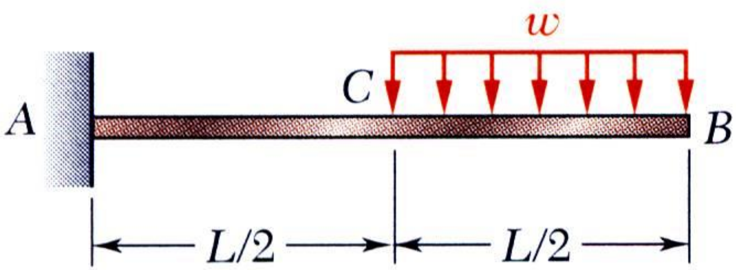
Method of Superposition



Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

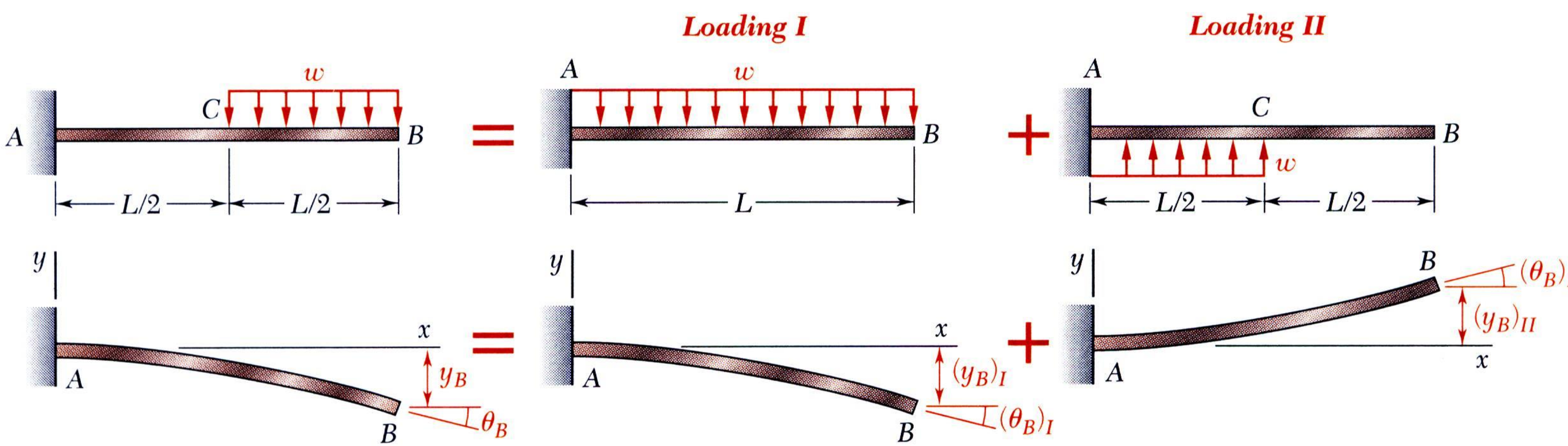
Sample Problem 9.7



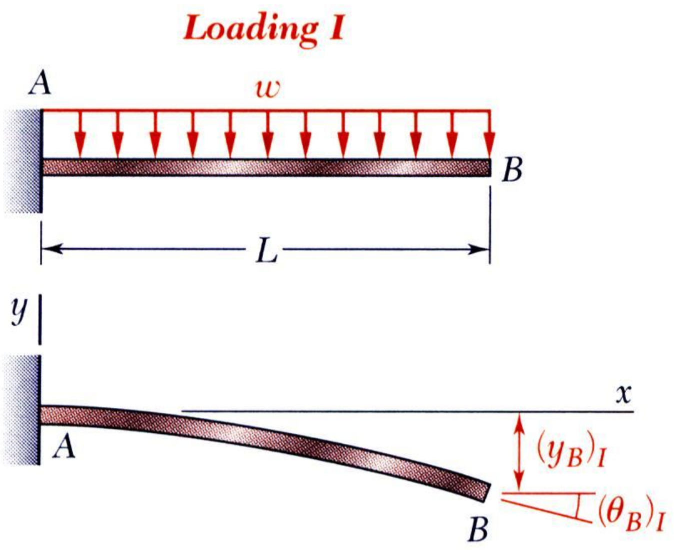
For the beam and loading shown, determine the slope and deflection at point *B*.

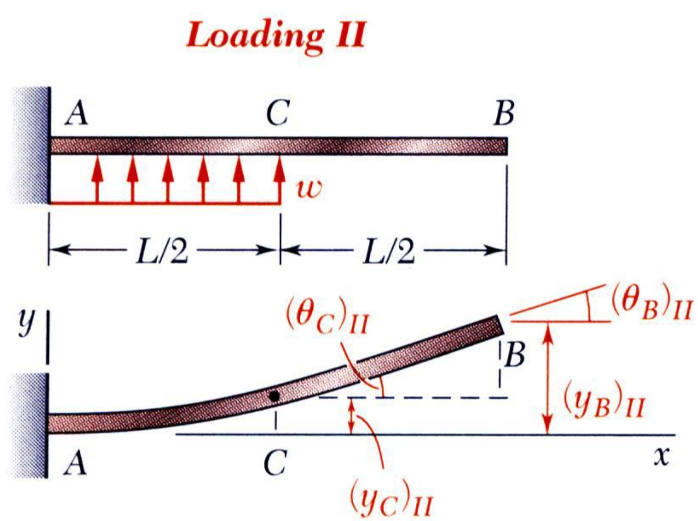
SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



Sample Problem 9.7





Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

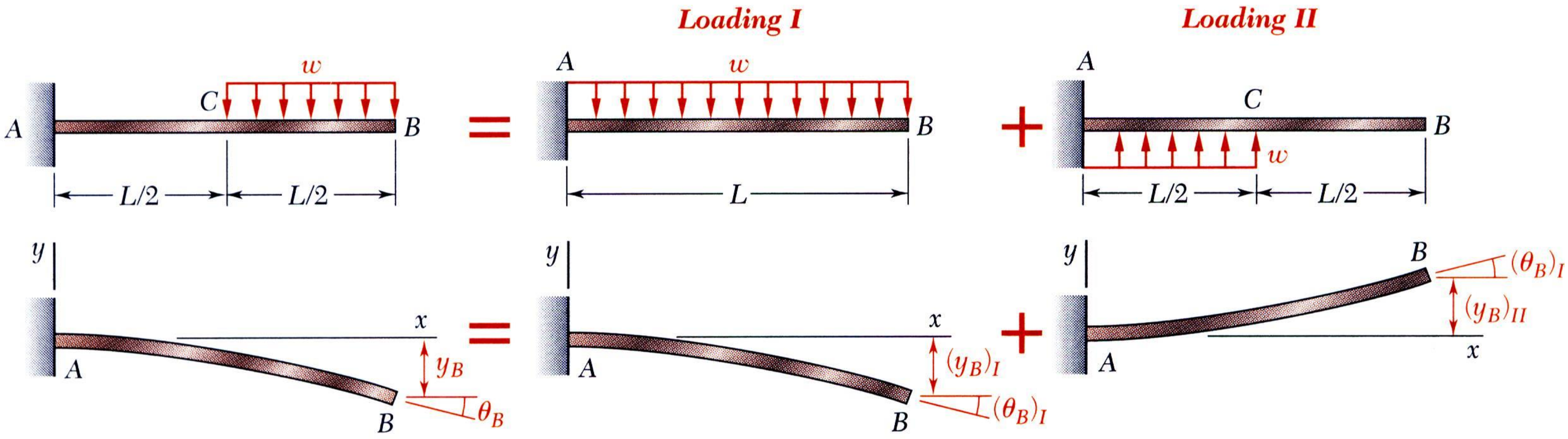
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$

In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

Sample Problem 9.7



Combine the two solutions,

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI}$$

$$\theta_B = -\frac{7wL^3}{48EI}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI}$$

$$y_B = -\frac{41wL^4}{384EI}$$